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Failure of Gauge Invariance in the Nonperturbative Formulation of Massless Lorentz-Violating QED

B. Altschul[1]

Department of Physics
Indiana University
Bloomington, IN 47405 USA

Abstract

We consider a Lorentz-violating modification to the fermionic Lagrangian of QED that is known to produce a finite Chern-Simons term at leading order. We compute the second order correction to the one-loop photon self-energy in the massless case using an exact propagator and a nonperturbative formulation of the theory. This nonperturbative theory assigns a definite value to the coefficient of the induced Chern-Simons term; however, we find that the theory fails to preserve gauge invariance at higher orders. We conclude that the specific nonperturbative value of the Chern-Simons coefficient has no special significance.

[1]`baltschu@indiana.edu`

There has been much recent interest in the possibility of adding CPT- and Lorentz-violating terms to the Lagrangians of quantum field theories [1, 2, 3, 4]. These terms may arise from violations of these symmetries at the Planck scale. There are many strong experimental constraints on Lorentz-violating corrections to the standard model, and such corrections must generally be small. However, the subject of Lorentz violation in quantum field theory is still of great theoretical and experimental interest.

The simplest perturbatively nontrivial correction to the fermion sector of quantum electrodynamics involves the addition of a CPT-violating axial vector term to the action. There has been a great deal of interest in the effect of such a term on the radiative corrections [4, 5, 6, 7, 8, 9, 10, 11] to the theory. At one loop order, the theory generates a term of the Chern-Simons form $\mathcal{L}_{CS} = \frac{1}{2}(k_{CS})_\mu \epsilon^{\mu\alpha\beta\gamma} F_{\alpha\beta} A_\gamma$ [12, 13, 14]. Astrophysical measurements constrain the physical coefficient $(k_{CS})_\mu$ to be very small [14, 15, 16]. However, while the radiatively induced value of $(k_{CS})_\mu$ is unambiguously finite, it is also completely undetermined (and possibly vanishing); its value depends upon how the theory is regulated. Moreover, since an arbitrary tree-level contribution may also be added to the radiatively induced term, the calculated value of the induced Chern-Simons coefficient can have no experimental significance.

There is, however, one particular value of the coefficient that appears to enjoy a special status. It is pointed out in [5] that if the theory is defined nonperturbatively in the CPT-violating axial vector interaction, then there is a precise value for the induced coefficient. This raises the question of whether this nonperturbative formulation has any special significance. In this paper, we answer that question, at least for the case of massless fermions, by analyzing higher-order contributions to the photon self-energy. We find that the nonperturbative formalism cannot be consistently applied to the calculation of the radiatively induced, CPT-even corrections to the theory. A nonperturbative regularization of the sort used in [5] leads unavoidably to a violation of the Ward identity that enforces the transversality of the vacuum polarization. Alternatively, while the use of dimensional regularization leads to unambiguously transverse CPT-even terms, this regularization restores the complete ambiguity in the CPT-odd terms, because γ_5 does not have a unique dimensional extension.

We shall introduce the massless theory and exhibit an extremely simple rationalization of the exact propagator. We then move on to calculate the one-loop photon self-energy $\Pi^{\mu\nu}(p)$. We shall interpret our results with the help of an analogy to a simpler theory, in which apparent violations of the Ward identity $p_\mu \Pi^{\mu\nu}(p) = 0$ can be eliminated by a change in the regularization; however, the corresponding change in regularization for the theory of interest is fundamentally nonperturbative. We conclude with a discussion of the implications of our result.

The Lagrange density for our theory (including a possible mass term) is

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - m - eA\!\!\!/ - b\!\!\!/\gamma_5)\psi. \tag{1}$$

However, we shall be concerned here only with the $m = 0$ case. Although we can eliminate

b from the massless Lagrangian by making the chiral transformation

$$\psi \to e^{-i\gamma_5 b\cdot x}\psi,\ \bar{\psi} \to \bar{\psi}e^{-i\gamma_5 b\cdot x}, \tag{2}$$

this transformation is anomalous and does not leave the gauge invariantly regulated fermionic measure invariant [17]. However, we should keep in mind that, if we account correctly for the anomaly associated with (2), we can eliminate b from the rest of the theory.

There are several reasons for considering only the massless case. Setting $m = 0$ simplifies the algebra in the calculation of the self-energy, but this is a minor point. There are two other, more important reasons. The first is the chiral symmetry mentioned above. The existence of this symmetry significantly simplifies our discussion; in particular, it allows us to construct a clear analogy that will illuminate the origin of the difficulties we encounter. The second reason is more subtle. We shall use power-counting arguments to determine the structure of the b-dependence of the vacuum polarization. In order to apply these arguments, we must suppose that the theory can be expanded in a power series in b. If there exists a nonvanishing momentum scale m^2 in the theory, it is conceivable that the power series description might break down at the scale $b^2 \sim m^2$. This actually occurs in the calculation of the Chern-Simons term [6, 7]. To avoid similar problems here, we set $m = 0$.

The exact fermion propagator for the massive theory is

$$S(k) = \frac{i}{\not{k} - m - \not{b}\gamma_5}. \tag{3}$$

When $m = 0$, this is most easily rationalized by breaking it into two terms, corresponding to the two eigenvalues of γ_5. Doing this, we have

$$\begin{aligned} S(k) &= \frac{i}{\not{k}-\not{b}}\frac{1-\gamma_5}{2} + \frac{i}{\not{k}+\not{b}}\frac{1+\gamma_5}{2} \\ &= i\left[\frac{\not{k}-\not{b}}{(k-b)^2}\frac{1-\gamma_5}{2} + \frac{\not{k}+\not{b}}{(k+b)^2}\frac{1+\gamma_5}{2}\right]. \end{aligned} \tag{4}$$

This rationalization of the propagator is substantially simpler than other versions. The numerators have fewer Dirac matrices, and the denominators are simpler as well. The presence of the right- and left-handed projectors will also simplify the algebra.

In particular, the one-loop self-energy,

$$\Pi^{\mu\nu}(p) = -ie^2 \mathrm{tr}\int \frac{d^4k}{(2\pi)^4}\gamma^\mu S(k)\gamma^\nu S(k+p), \tag{5}$$

may be simplified in the massless case to

$$\begin{aligned} \Pi^{\mu\nu}(p) &= \frac{ie^2}{2}\mathrm{tr}\int \frac{d^4k}{(2\pi)^4}\left[\frac{\gamma^\mu(\not{k}+\not{b})\gamma^\nu(\not{k}+\not{p}+\not{b}) + \gamma^\mu(\not{k}+\not{b})\gamma^\nu(\not{k}+\not{p}+\not{b})\gamma_5}{(k+b)^2(k+p+b)^2}\right. \\ &\quad \left. + \frac{\gamma^\mu(\not{k}-\not{b})\gamma^\nu(\not{k}+\not{p}-\not{b}) - \gamma^\mu(\not{k}-\not{b})\gamma^\nu(\not{k}+\not{p}-\not{b})\gamma_5}{(k-b)^2(k+p-b)^2}\right]. \end{aligned} \tag{6}$$

Through an analysis of the structure of (6), we may learn a great deal about the nature of the nonperturbative theory.

We observe that all the information necessary for the calculation of $\Pi^{\mu\nu}(p)$ is contained in the function

$$f_{\alpha\beta}(p,b)=\int\frac{d^4k}{(2\pi)^4}\frac{(k-b)_\alpha(k+p-b)_\beta}{(k-b)^2(k+p-b)^2}. \tag{7}$$

The self-energy involves symmetric and antisymmetric sums $f_{\alpha\beta}(p,b)\pm f_{\alpha\beta}(p,-b)$ contracted with tensors in (μ,ν,α,β). The terms of $f_{\alpha\beta}(p,b)$ that are even in b give rise to contributions to $\Pi^{\mu\nu}(p)$ with different Lorentz structure than the terms that are odd in b, since the odd terms involve a trace over γ_5. We might conclude that, because of the differences in their Lorentz structures, the two types of terms will need to be regulated differently. However, as a formal object, $f_{\alpha\beta}(p,b)$ still contains everything needed to determine the one-loop self-energy.

In fact, the nonperturbative viewpoint requires that the same regulator be used for all the terms, regardless of whether they are even or odd in b. Since there is only a single Feynman diagram in the nonperturbative formulation, a truly nonperturbative calculation would involve a single evaluation of $f_{\alpha\beta}(p,b)$ to all orders in b using a unique regularization prescription. It would not be consistent to use the nonperturbative regularization for the b-odd terms and a different regulator for the even terms. (It is possible that in a more fundamental theory, we may be required to use a regulator that does treat the even and odd terms differently; however, this is just speculation.) We shall therefore use the same methods used in [5] to fix the coefficient of the induced Chern-Simons term to determine the higher-order, CPT-even contributions to the self-energy.

If we shift the integration variable $k\rightarrow k-b$ in $f_{\alpha\beta}(p,b)$, then the integrand becomes b-independent. Since the integral is superficially quadratically divergent, the surface term generated by the shift is at most quadratic in b. Therefore, there are no contributions to the self-energy that are higher than second order in b. This result has been previously demonstrated for the b-odd terms, and it might be expected on dimensional grounds for the b-even terms as well, since there is no mass scale in the problem. However, the ease with which it has been demonstrated here shows the usefulness of the propagator (4).

The $\mathcal{O}(b^0)$ part of $f_{\alpha\beta}(p,b)$ gives the usual QED photon self-energy. The $\mathcal{O}(b)$ contribution to $\Pi^{\mu\nu}(p)$ has also been calculated; this is the Chern-Simons term. In a specific nonperturbative formalism, which is presented in [5, 6, 7], this term has a fixed value, but more generally—in particular, if the theory is defined perturbatively in b—its value is undetermined and regularization-dependent [10]. However, this ambiguity and the structure of this term in general are well understood. We shall therefore focus our attention on the $\mathcal{O}(b^2)$ terms.

If we were interested only in finding the ultraviolet divergent part of $f_{\alpha\beta}(p,b)$ at second order in b, we could simply shift the integration as outlined above and evaluate the resulting integral; since the initial integral is superficially quadratically divergent, the $\mathcal{O}(b^2)$ part of the surface term accompanying the shift is ultraviolet finite. We would find

that the ultraviolet divergent part of $f_{\alpha\beta}(p,b)$ vanishes at $\mathcal{O}(b^2)$. However, if we wish to calculate the ultraviolet finite contributions (which are derived entirely from the surface term accompanying the integration shift) as well, we must take more care.

We shall evaluate the $\mathcal{O}(b^2)$ terms in $f_{\alpha\beta}(p,b)$ by a direct expansion of the integrand in powers of b. The integral that appears at second order in b is finite when the integration is performed symmetrically. This is the correct prescription for performing the integration in the nonperturbative formalism, because the same property (observer Lorentz invariance) is being used to fix the value of the vacuum polarization at both $\mathcal{O}(b)$ and $\mathcal{O}(b^2)$. [Of course, the same technique cannot be used to deal with the $\mathcal{O}(b^0)$ term in the photon self-energy if gauge invariance is to be preserved. We shall set this formal difficulty aside, however, since the $\mathcal{O}(b^0)$ is necessarily divergent in any regularization scheme and is thus qualitatively different from the higher-order terms.]

We begin our calculation by writing

$$f_{\alpha\beta}(p,b)=\int\frac{d^4k}{(2\pi)^4}\exp\left(-b_\gamma\frac{\partial}{\partial k_\gamma}\right)\frac{k_\alpha}{k^2}\frac{k'_\beta}{(k')^2}, \tag{8}$$

where we have defined $k'=k+p$. The portion of this expression that is quadratic in b, which we shall denote as $h_{\alpha\beta}(p,b)$, is

$$\begin{aligned} h_{\alpha\beta}(p,b) &= \frac{1}{2}b_\gamma b_\delta\int\frac{d^4k}{(2\pi)^4}\frac{\partial}{\partial k_\gamma}\frac{\partial}{\partial k_\delta}\frac{k_\alpha}{k^2}\frac{k'_\beta}{(k')^2} && (9)\\ &= \frac{1}{96\pi^2}(b_\alpha b_\beta-g_{\alpha\beta}b^2). && (10)\end{aligned}$$

Transversality of the vacuum polarization requires that $(2p^\alpha g^{\nu\beta}-p^\nu g^{\alpha\beta})h_{\alpha\beta}(p,b)=0$. This condition does not hold; even though $h_{\alpha\beta}(p,b)$ is unambiguously finite, it still violates the Ward identity. Gauge invariance is broken, so the theory becomes nonrenormalizable, and we encounter divergences, which render the theory undefined.

To understand how this violation of gauge invariance arises, it is useful to consider an analogy. For a theory with Lagrange density

$$\mathcal{L}'=-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}+\bar{\psi}(i\partial\!\!\!/-eA\!\!\!/-a\!\!\!/)\psi, \tag{11}$$

there is a field rescaling similar to (2),

$$\psi\rightarrow e^{-ia\cdot x}\psi,\ \bar{\psi}\rightarrow e^{ia\cdot x}\bar{\psi}, \tag{12}$$

which eliminates a from the theory. Moreover, this rescaling is not anomalous. However, we may choose not to eliminate a from the Lagrangian. If we consider the theory directly as defined by $\mathcal{L}'$ and attempt to calculate the photon self-energy, we will immediately be led to an evaluation of $f_{\alpha\beta}(p,a)$. The violation of the Ward identity in this instance is

clearly an artifact of our unconventional choice of momentum coordinates. A shift in the integration variable $k \rightarrow k - a$ eliminates the problematic term. This shift is precisely equivalent to the field redefinition (12).

Returning to the axial vector theory (1), it seems now that the correct solution to our difficulties would be to shift the origin of the integration in (7) so that $h_{\alpha\beta}(p, b)$ is set to zero. While we believe that this is the physically correct way of regulating this theory, it is inconsistent with the nonperturbative formalism. In the vector theory with a, there arises in the calculation of $\Pi^{\mu\nu}(p)$ only a single term of the form $f_{\alpha\beta}(p, a)$. However, in the axial vector theory, we encounter the sum $f_{\alpha\beta}(p, b) + f_{\alpha\beta}(p, -b)$. The essence of the nonperturbative formulation is that both terms in (6) must be regulated in the same way. We are allowed only a single shift in the integration variable $k \rightarrow k + q$. This shift transforms $f_{\alpha\beta}(p, b) + f_{\alpha\beta}(p, -b) \rightarrow f_{\alpha\beta}(p, q + b) + f_{\alpha\beta}(p, q - b) \neq 0$. We see that in order to eliminate the Ward-identity-violating surface term, we must be free to shift the integrations in different terms by different amounts, which is equivalent to defining the theory perturbatively. [Although the theory is massless, a chiral shift $k \rightarrow k + \gamma_5 q$ cannot be implemented in a fashion that allows us to retain a unique result at $\mathcal{O}(b)$, for reasons we shall outline below.] In the perturbative formulation, we treat the term $-\bar{\psi}\, \not{b} \gamma_5 \psi$ in $\mathcal{L}$ as defining a new vertex of the theory. We are then free to shift the integrations independently in the evaluations of different Feynman diagrams, and this allows us to enforce gauge invariance.

Although we are working with a massless theory, we are not allowed to make naive chiral shifts in the integration variable k if we are to retain the special nonperturbative value for $(k_{CS})_\mu$, because the corresponding transformation (2) is anomalous. To properly account for the anomaly, we must use the functional integral formalism. The Fujikawa determinant [17] accompanying the shift then reproduces the correct nonperturbative value of the induced Chern-Simons coefficient; however, there is another, completely undetermined contribution that arises from the ambiguity in the definition of the axial current operator [18]. So it is impossible to shift the integrations separately for the left- and right-handed components of ψ without giving up the uniqueness of the $\mathcal{O}(b)$ result.

It is the necessity of using a single regulator for both terms in (6) that gives rise to a unique specification of the induced Chern-Simons term at $\mathcal{O}(b)$. However, it proved impossible to regulate the $\mathcal{O}(b^2)$ terms in the same fashion without violating the Ward identity. Therefore, there seems to be no special significance to the nonperturbative value of the induced Chern-Simons term. This is in some ways unsurprising. Although there are stability problems for theories with nonvanishing Chern-Simons coefficients, we are always free to add an additional Chern-Simons term to the bare Lagrangian so as to make the total coefficient zero.

The existence of a surface term that violates gauge invariance at $\mathcal{O}(b^2)$ is perhaps also unsurprising, given what is known about the behavior of the nonperturbative theory at $\mathcal{O}(b)$. In the massless case, the only contribution to the induced Chern-Simons coefficient comes from the surface term. The associated induced Lagrange density is not gauge

invariant; however, because the density necessarily involves $\epsilon^{\mu\nu\alpha\beta}$, the Ward identity is preserved and the integrated action remains gauge invariant. There is no such restriction on the form of the induced term at $\mathcal{O}(b^2)$, and without the protection of a specific structure for the self-energy, the gauge invariance of the action is lost. At each order, the surface term simply violates gauge invariance in the strongest way allowed by its tensor structure.

Finally, we must discuss the possibility of dimensional regularization. A dimensional regulator preserves gauge invariance at all orders in b and sets $h_{\alpha\beta}(p,b)=0$. Moreover, it solves the formal problems associated with the nonperturbative evaluation of the $\mathcal{O}(b^0)$ contributions, since it allows us to regulate all the terms, even the divergent ones, in the same fashion. However, dimensional regularization also restores the complete ambiguity of the induced Chern-Simons term. (This is closely analogous to the situation we encountered when using the functional integral formalism and a Fujikawa regulator.) The b-odd terms involve γ_5, which does not have a unique extension to $4-\epsilon$ dimensions. Any extension that commutes with $4-n\epsilon$ γ-matrices (for arbitrary n) will have the correct limit as $\epsilon\rightarrow 0$, and each extension will give a different result for the Chern-Simons coefficient. Moreover, it is not possible to determine the correct extension from other sectors of the theory. The dimensional extension of the γ_5 appearing in $-\bar{\psi}\,\not{b}\gamma_5\psi$ need not be the same as the dimensional extensions of the γ_5 appearing in the chiral gauge sector, for example; these are entirely distinct operators, which may behave differently under dimensional regularization. Therefore, while a dimensional regularization prescription may be used to implement a completely nonperturbative formulation of the theory, it also renders the coefficient of the induced Chern-Simons term completely undetermined.

In this paper, we have demonstrated that the ambiguity in the value of the induced Chern-Simons term for the theory defined by (1) is unavoidable. The specific value predicted by the nonperturbative formulation can have no special significance, because the nonperturbative regularization of the theory used in [5] cannot be extended to higher orders in b without violating the Ward identity; and therefore, even if it contains no tree-level contribution, the coefficient of Chern-Simons term in the effective action can be fixed only by experiment. Although this result was derived only in the limit of massless fermions, the massive theory may well behave similarly.

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